Technical Appendix: Specification of the CP2022 Model

The model CP2022 that has been used in 2022 by the Commissie Parameters to generate scenarios under equivalent measures $\mathbb{P}$ and $\mathbb{Q}$ is an affine model with stochastic volatility. Affine models ensure that the value of certain financial contracts can be expressed explicitly in terms of a limited number of parameters. We refer to the book of Filipovic [5] for an overview of such models and their properties. The model CP2022 is an extension of the model used in 2019 by the previous Commissie Parameters, which was based on the KNW model [8] and subsequent modifications and analysis by Draper [3] and Muns [9]. Other relevant papers include those by Brennan & Xia [2], Duffie & Kan [4], Singor et. al [10], Schöbel & Zhu [11] and van Haastrecht & Pelsser [12].

1 State Equations

The economic model CP2022 is based on a stochastic process $\mathbf{X}$:

$$\mathbf{X}_t = \begin{bmatrix} \mathbf{X}_t^s \\ \mathbf{X}_t^o \end{bmatrix}, \qquad \mathbf{X}_t^s = \begin{bmatrix} v_t \\ r_t \\ \pi_t \end{bmatrix}, \qquad \mathbf{X}_t^o = \begin{bmatrix} \ln(S_t) \\ \ln(\Pi_t) \end{bmatrix}, \tag{1}$$

which consists of a state vector process $\mathbf{X}^s$, which contains a short rate process r, an expected (European) inflation rate process π and a stochastic variance process v (see Heston [6]) which equals the square of the stochastic volatility process. The two additional variables are the logarithm of a stock price index S and (European) consumer price index Π, which together form $\mathbf{X}^o$. Dutch consumer prices will be modeled separately later in this Appendix.

The dynamics of $\mathbf{X}^s$ is described by

$$\begin{aligned} d\mathbf{X}_t^s &= \begin{bmatrix} K_{vv} & 0 & 0 \\ K_{vr} & K_{rr} & K_{\pi r} \\ K_{v\pi} & K_{r\pi} & K_{\pi\pi} \end{bmatrix} \left(\begin{bmatrix} \mathbb{E}v_\infty \\ \mathbb{E}r_\infty \\ \mathbb{E}\pi_\infty \end{bmatrix} - \mathbf{X}_t^s \right) dt \;+\; \begin{bmatrix} \omega & 0 & 0 & 0 & 0 \\ \sigma_{vr} & \sigma_{r1} & \sigma_{r2} & 0 & 0 \\ \sigma_{v\pi} & \sigma_{\pi 1} & \sigma_{\pi 2} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + v_t\Gamma_1 \end{bmatrix}^{\frac{1}{2}} dW_t^{\mathbb{P}}, \\ &=: \; K(\mathbb{E}\mathbf{X}_\infty^s - \mathbf{X}_t^s)\, dt \;+\; \Sigma^{r\pi}(\Gamma_0 + (\mathbf{X}_t^s)_1\Gamma)^{\frac{1}{2}}\, dW_t^{\mathbb{P}}, \end{aligned} \tag{2}$$

where $A^{\frac{1}{2}}$ denotes[1] the symmetric matrix H satisfying $HH' = H^2 = A$, $W_t^{\mathbb{P}}$ is a 5-dimensional standard Brownian Motion (with independent components) and

$$\mathbb{E}\mathbf{X}_\infty^s = \begin{bmatrix} \mathbb{E}v_\infty \\ \mathbb{E}r_\infty \\ \mathbb{E}\pi_\infty \end{bmatrix},\; K = \begin{bmatrix} K_{vv} & 0 & 0 \\ K_{vr} & K_{rr} & K_{\pi r} \\ K_{v\pi} & K_{r\pi} & K_{\pi\pi} \end{bmatrix},\; \Gamma = \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & \Gamma_1 \end{bmatrix},\; \Gamma_0 = \begin{bmatrix} 0 & 0_{1\times 4} \\ 0_{4\times 1} & I_4 \end{bmatrix},\; \Sigma^{r\pi} = \begin{bmatrix} \omega & 0 & 0 & 0 & 0 \\ \sigma_{vr} & \sigma_{r1} & \sigma_{r2} & 0 & 0 \\ \sigma_{v\pi} & \sigma_{\pi 1} & \sigma_{\pi 2} & 0 & 0 \end{bmatrix}. \tag{3}$$

We impose that (i) $\omega \geq 0$, (ii) K and Γ_1 have real positive eigenvalues and (iii) Γ_1 has zero values outside its diagonal[2]. To ensure that $v_0 > 0$ implies $\mathbb{P}(v_t > 0) = 1$ we also impose the Feller condition $K_{vv}\mathbb{E}v_\infty - \frac{1}{2}\omega^2 \geq 0$.

The logarithm of indices for stock prices and (European) consumer prices in $\mathbf{X}^o$ satisfy

$$\begin{aligned} d\mathbf{X}_t^o &= \begin{bmatrix} r_t + \eta_S \\ \pi_t + \eta_\Pi \end{bmatrix} dt \;-\; \tfrac{1}{2}\mathcal{D}\left(\begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix} \begin{bmatrix} v_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + v_t\Gamma_1 \end{bmatrix} \begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}' \right) dt \;+\; \begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix} \begin{bmatrix} v_t & 0_{1\times 4} \\ 0_{4\times 1} & I_4 + v_t\Gamma_1 \end{bmatrix}^{\frac{1}{2}} dW_t^{\mathbb{P}}, \\ &=: \; (\mu^o + K^o\mathbf{X}_t^s)dt \;+\; \Sigma^{S\Pi}(\Gamma_0 + (\mathbf{X}_t^s)_1\Gamma)^{\frac{1}{2}}\, dW_t^{\mathbb{P}}, \end{aligned} \tag{4}$$

with η_S and η_Π in $\mathbb{R}$, and σ_S and σ_Π both vectors in $\mathbb{R}^5$. We use the symbol $\mathcal{D}(A)$ for the diagonal of a matrix A, expressed as a column vector, and define

$$\mu^o = \begin{bmatrix} \eta_S \\ \eta_\Pi \end{bmatrix} - \tfrac{1}{2}\mathcal{D}(\Sigma^{S\Pi}\Gamma_0\Sigma^{S\Pi\prime}), \qquad \Sigma^{S\Pi} = \begin{bmatrix} \sigma_S' \\ \sigma_\Pi' \end{bmatrix}, \tag{5}$$

$$K^o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tfrac{1}{2}\mathcal{D}(\Sigma^{S\Pi}\,\Gamma\Sigma^{S\Pi\prime})\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{6}$$

We impose that $(\sigma_\Pi)_4 = 0$; this and other zero values in the specification of matrices and vectors have been chosen in order to make the model specification unique.

[1] We choose a symmetric form for H to facilitate interpretation; one can also use a Cholesky representation.

[2] We have not chosen for a further extension of the previous model in which Γ_1 can have off-diagonal elements equal to zero, to keep the model parsimonious.

2 Market prices of risk

Market prices of risk which characterize the transformation from $\mathbb{P}$ to $\mathbb{Q}$ are affine in $\mathbf{X}^s$ and defined by a 5×3 matrix Λ_1 and a 5-dimensional vector λ_0:

$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - ((\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}})^{-1} (\lambda_0 + \Lambda_1 \mathbf{X}_t^s) \, dt, \tag{7}$$

which should satisfy the constraints:

$$\Sigma^{S\Pi} \lambda_0 = \left[\begin{smallmatrix} \eta_S \\ \eta_\Pi \end{smallmatrix} \right], \tag{8}$$

$$\Sigma^{S\Pi} \Lambda_1 = 0_{2 \times 3}. \tag{9}$$

We note that the risk premium on inflation risk η_Π is zero in the original KNW paper; this constraint has not been incorporated in CP2022.

We define

$$M = K + \Sigma^{r\pi} \Lambda_1 \tag{10}$$

for the riskneutral version of K and impose $(\Lambda_1)_{1,2} = (\Lambda_1)_{1,3} = 0$ to ensure that K and M have the same zero elements imposed. The eigenvalues of M are required to be real and positive and we impose the Feller condition $M_{vv} \mathbb{E}^{\mathbb{Q}} v_\infty - \frac{1}{2}\omega^2 \geq 0$, to ensure that $\mathbb{Q}(v_t > 0) = 1$.

To create a riskneutral version of $\mathbb{E}\mathbf{X}_\infty^s$, i.e. $\mathbb{E}^{\mathbb{Q}}\mathbf{X}_\infty^s$, in

$$d\mathbf{X}_t^s = M \left(\mathbb{E}^{\mathbb{Q}} \mathbf{X}_\infty^s - \mathbf{X}_t^s \right) \, dt + \Sigma^{r\pi} (\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}} \, dW_t^{\mathbb{Q}}, \tag{11}$$

$$d\mathbf{X}_t^o = (\mu^o - \left[\begin{smallmatrix} \eta_S \\ \eta_\Pi \end{smallmatrix} \right] + K^o \mathbf{X}_t^s) \, dt + \Sigma^{S\Pi} (\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}} \, dW_t^{\mathbb{Q}}. \tag{12}$$

we must choose

$$\Sigma^{r\pi} \lambda_0 = -M \, \mathbb{E}^{\mathbb{Q}} \mathbf{X}_\infty^s + K \, \mathbb{E} \mathbf{X}_\infty^s. \tag{13}$$

Choosing K, M, $\mathbb{E}\mathbf{X}_\infty^s$ and $\mathbb{E}^{\mathbb{Q}}\mathbf{X}_\infty^s$ fixes λ_0 and Λ_1 using (8), (9), (10) and (13).

3 Term structures of interest

The yield of a nominal zero coupon bond at time t with time to maturity τ (i.e. with a payoff of one euro at time $t + \tau$) satisfies

$$\begin{aligned} y_t(\tau) &= -\tau^{-1} \ln \mathbb{E}_t^{\mathbb{Q}} e^{-\int_t^{t+\tau} r_u du} = -\tau^{-1} \ln \mathbb{E}_t^{\mathbb{Q}} e^{[0 \ -1 \ 0] \int_t^{t+\tau} \mathbf{X}_u^s du} \\ &= -\tau^{-1} (\phi(t, t+\tau) + \Psi(t, t+\tau)' \mathbf{X}_t^s), \end{aligned} \tag{14}$$

for deterministic functions ϕ and Ψ that solve the Riccati equations given in section 10, if we substitute the following input parameters in those equations to characterize the dynamics under $\mathbb{Q}$:

$$L = M, \quad \zeta(t) = M \mathbb{E}^{\mathbb{Q}} \mathbf{X}_\infty^s, \quad \Sigma = \Sigma^{r\pi}, \quad u' = [0 \ 0 \ 0], \quad v' = [0 \ -1 \ 0], \tag{15}$$

with $G_0 = \Gamma_0$, $G_1 = \Gamma$, and G_i the zero matrix with the same dimensions as G_0 for $i > 1$.

Real yields will satisfy

$$\begin{aligned} y_t^R(\tau) &= -\tau^{-1} \ln \mathbb{E}_t^{\mathbb{Q}} e^{-\int_t^{t+\tau} r_u du + (\ln \Pi_{t+\tau} - \ln \Pi_t)} = -\tau^{-1} \ln \mathbb{E}_t^{\mathbb{Q}} e^{\int_t^{t+\tau} [0 \ -1 \ 0] \mathbf{X}_u^s du + [0 \ 1] (\mathbf{X}_{t+\tau}^o - \mathbf{X}_t^o)} \\ &= -\tau^{-1} (\ \phi_R(t, t+\tau) + \Psi_R^s(t, t+\tau)' \mathbf{X}_t^s + (\Psi_R^o(t, t+\tau)' - [0 \ 1]) \mathbf{X}_t^o \ ), \end{aligned} \tag{16}$$

for deterministic functions ϕ_R, Ψ_R^s and Ψ_R^o that solve the riskneutral version of the Riccati equations in section 10 for the process $\mathbf{X}$ in (1) which combines $\mathbf{X}^s$ and $\mathbf{X}^o$:

$$L = \begin{bmatrix} M & 0_{3 \times 2} \\ -K^0 & 0_{2 \times 2} \end{bmatrix}, \qquad \zeta(t) = \begin{bmatrix} M \mathbb{E}^{\mathbb{Q}} \mathbf{X}_\infty^s \\ \mu^o - \left[\begin{smallmatrix} \eta_S \\ \eta_\Pi \end{smallmatrix} \right] \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix}, \tag{17}$$

with the same G_0, G_1, and G_i (for $i > 1$) as above and

$$u' = [0\ 0\ 0\ 0\ 1], \quad v' = [0\ -1\ 0\ 0\ 0]. \tag{18}$$

Since the last two rows of L' and v are zero and G_4 and G_5 are zero matrices, we see from (78) in section 10 that $\Psi_R^o(t, t+\tau)' = [0\ 1]$ so the term involving $\mathbf{X}_t^o$ in (16) disappears and we may remove the superscript in Ψ_R^s:

$$y_t^R(\tau) \quad = \quad -\tau^{-1}(\ \phi_R(t, t+\tau) + \Psi_R(t, t+\tau)'\mathbf{X}_t^s\ ). \tag{19}$$

In our model specification we can write, by a slight abuse of notation, that $\Psi(t, t+\tau) = \Psi(\tau)$ and $\Psi^R(t, t+\tau) = \Psi^R(\tau)$ and as long as $\zeta(t)$ is constant we can also write $\phi_R(t, t+\tau) = \phi_R(\tau)$.

4 Historical observations and calibration under $\mathbb{P}$

We assume that the price indices S_t, Π_t, the squared volatility v_t and the yields of one nominal and one real bond, with maturities τ_N^* and τ_R^* respectively, can be observed without measurement error [3, 7]. This implies that we can indirectly observe the state $(\mathbf{X}_t^s, \mathbf{X}_t^o) = (v_t, r_t, \pi_t, \ln S_t, \ln \Pi_t)$ since

$$\mathbf{X}_t^{s,\text{obs}} \quad = \quad \begin{bmatrix} 1\ 0\ 0 \\ \Psi(t,t+\tau_N^*)' \\ \Psi_R(t,t+\tau_R*)' \end{bmatrix}^{-1} \begin{bmatrix} v_t^{\text{obs}} \\ -\tau_N^* y_t(\tau_N^*)^{\text{obs}} - \phi(t,t+\tau_N^*) \\ -\tau_R^* y_t(\tau_R^*)^{\text{obs}} - \phi(t,t+\tau_R^*) \end{bmatrix}. \tag{20}$$

For some other maturities we assume that nominal and real yields can be observed with measurement errors, and that these have been collected in vector processes $\mathbf{Y}_t^{\text{obs}}$ and $\mathbf{Y}_t^{R,\text{obs}}$ of length n_y and n_y^R respectively. The corresponding maturity vectors are τ and τ^R, and the standard deviations of measurement equation errors together form a vector h. We can then characterize the measurement equation errors ϵ_t^y by

$$\begin{bmatrix} \mathbf{Y}_t^{\text{obs}}(\tau_1) \\ \vdots \\ \mathbf{Y}_t^{\text{obs}}(\tau_{n_y}) \\ \mathbf{Y}_t^{\text{R,obs}}(\tau_1^R) \\ \vdots \\ \mathbf{Y}_t^{\text{R,obs}}(\tau_{n_y^R}^R) \end{bmatrix} \quad = \quad \begin{bmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_{n_y}) \\ y_t^R(\tau_1^R) \\ \vdots \\ y_t^R(\tau_{n_y^R}^R) \end{bmatrix} + \epsilon_t^y, \qquad \epsilon_t^y \sim N(0_{(n_y+n_y^R)\times 1},\ \Sigma^y),\ \text{iid} \tag{21}$$

with $\Sigma^y \in \mathbb{R}^{(n_y+n_y^R)\times(n_y+n_y^R)}$ a matrix with zero elements apart from the diagonal, which contains values h_i^2 for $1 \le i \le n_y + n_y^R$. Model error processes ϵ_t^{so} for given parameters can be approximated by (see (2) and (4)):

$$\begin{bmatrix} \mathbf{X}_{t+\Delta t}^{s,\text{obs}} - \mathbf{X}_t^{s,\text{obs}} \\ \mathbf{X}_{t+\Delta t}^{o,\text{obs}} - \mathbf{X}_t^{o,\text{obs}} \end{bmatrix} \quad = \quad \begin{bmatrix} K(\mathbb{E}\mathbf{X}_\infty^s - \mathbf{X}_t^{s,\text{obs}}) \\ \mu^o + K^o \mathbf{X}_t^{s,\text{obs}} \end{bmatrix} \Delta t + \epsilon_t^{\text{so}}, \tag{22}$$

with

$$\epsilon_t^{\text{so}} \sim N(0_{5\times 1}, \Sigma_t^{\text{so}})\ \text{iid}, \qquad (\Sigma_t^{\text{so}})^{\frac{1}{2}} \quad = \quad \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} (\Gamma_0 + (\mathbf{X}_t^{s,\text{obs}})_1 \Gamma)^{\frac{1}{2}} \sqrt{\Delta t}. \tag{23}$$

The processes ϵ^{so} are assumed to be independent from ϵ^y. This means that for the log-likelihood optimization under $\mathbb{P}$ we need to maximize[3]

$$\ln L^{\mathbb{P}} \quad = \quad -\tfrac{1}{2} \sum_{t=1}^{n} \Big(\ \epsilon_t^{y\prime}(\Sigma^y)^{-1}\epsilon_t^y \ + \ \epsilon_t^{\text{so}\prime}(\Sigma_t^{\text{so}})^{-1}\epsilon_t^{\text{so}} \ + \ \ln\det(\Sigma^y) \ + \ \ln\det(\Sigma_t^{\text{so}})\ \Big) + c \tag{24}$$

[3]Note that we use an equal number of ϵ^y and ϵ^{so} values. We have one observation more for the ϵ^y-values (since these do not involve taking differences), but we do not use the first value of ϵ^y.

over parameters $\Theta = (\mathbb{E}\mathbf{X}^s_\infty, K, \Gamma_1, \Sigma^{r\pi}, \Sigma^{S\Pi}, \eta_S, \eta_\Pi, h, \mathbb{E}^{\mathbb{Q}}\mathbf{X}^s_\infty, M)$, with c a constant that does not need to be included in the optimization. The last two parameters enter the likelihood optimization through the functions y_t and y^R_t since the Riccati equations that they solve depend on these riskneutral parameters. For the calibration of CP2022 we use $\{\tau_1, ..\tau_5\} = \{1,\ 5,\ 10,\ 20,\ 30\}$ and $\tau^*_N = 15$, and $\{\tau^R_1, ..\tau^R_5\} = \{1,\ 5,\ 15,\ 20,\ 30\}$ and $\tau^*_R = 10$.

5 Market data and calibration under $\mathbb{P}$ and $\mathbb{Q}$

To jointly estimate model parameters for the dynamics under $\mathbb{P}$ and $\mathbb{Q}$, we optimize the goal function $\ln L^{\mathbb{P}}(\Theta)$ defined above under extra constraints that are based on observed market data at the time of calibration t_0. These concern the squared relative difference between implied volatilities generated by model parameters and observed implied volatilities of financial derivatives at the time of calibration. We therefore impose

$$e_{\text{eq}}(\Theta)^2 \leq (1.50\%)^2, \qquad e_{\text{int}}(\Theta)^2 \leq (0.15\%)^2, \qquad e_{\text{infl}}(\Theta)^2 \leq (0.50\%)^2,$$

with

$$e_{\text{class}}(\Theta)^2 = \frac{1}{n_{\text{class}}} \sum_{k=1}^{n_{\text{class}}} \left(\frac{p_k^{\text{class,observed}} - p_k^{\text{class,model}}(\Theta)}{\mho_k^{\text{class}}} \right)^2, \tag{25}$$

for class $\in \{\text{eq}, \text{int}, \text{infl}\}$. In this expression, $p_k^{\text{class,observed}}$ denotes the observed market price of the k-th instrument in one of the three derivative classes (equity derivatives, interest rate derivatives and inflation derivatives), $p_k^{\text{class,model}}(\Theta)$ is the corresponding price implied by the model for a choice of parameters Θ and $\mho_k^{\text{class}}$ is the vega of the k-th instrument. Prices are determined using simulations under $\mathbb{Q}$ that will be specified in section 9 of this Appendix. We give closed-form expressions for the vega values in section 12.

By dividing the difference in prices by the corresponding vegas, we approximate differences in implied volatilities. By squaring these, we obtain an approximation of the instruments' relative quadratic error in terms of implied volatilities.

6 Exact fitting of the term structure

Let t_0 denote the time of calibration for the model. To fit the term structure we use a market price of risk which is assumed to be a constant λ_0 at all times $t < t_0$ for the calibration of historical asset prices, but assumed to be time-varying[4] for all future times $t \geq t_0$.

In simulations we use monthly time steps $\Delta = \frac{1}{12}$ so we use monthly nominal and real yields $y^{\text{obs}}_{i\Delta}(t_0)$ and $y^{\text{R,obs}}_{i\Delta}(t_0)$ to fit the curve. We have nominal bond observations for yearly maturities $\{1, 2, 3, ..., 50\}$ and real bond observations for maturities $\{1, 2, ..., 9, 10, 12, 15, 20, 25, 30, 40, 50\}$. Where needed, we use piecewise linear interpolation of the function $\tau \to \ln p_{t_0}(\tau)$ to obtain intermediate values between observations. Since this function takes the value 0 for $\tau = 0$ this allows extrapolation for maturities before the maturity of one year as well[5]. We extrapolate for maturities above maturity 50 by making the yearly forward rates after that maturity equal to the yearly forward rate between maturities 30 and 50; we do this both for the nominal and the real curve.

We define[6]

$$\tilde{\lambda}_0(t) = \lambda_0 + \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix}^{-1} [0\ f(t)\ 0\ 0\ f^R(t)]' \mathbf{1}_{\{t \geq t_0\}}, \tag{26}$$

[4]Note that we do not incorporate the fact that λ_0 becomes time-varying after the calibration time t_0 when determining the historical bond prices that are used in the calibration.

[5]The term structures that we fit are characterized by piecewise constant forward rates. The shift functions f and f^R that we define below will be smoother when term structures with smoother forward rates are used in the calibration.

[6]We remark that the restriction $(\Sigma\lambda_0)_4 = \eta_s$ in (8) and our choice of $\tilde{\lambda}_0(t)$ ensure that $(\Sigma\tilde{\lambda}_0(t))_4 = \eta_s$ as well.

with $f(t)$ and $f^R(t_0+\tau)$ equal to constants f_i and f_i^R for $\tau \in [i\Delta, (i+1)\Delta[$ so $f(t_0+\tau) = \sum_{i=0}^{\infty} f_i \mathbf{1}_{\{\tau\in[i\Delta,\,(i+1)\Delta[\}}$ and a similar equation holds for $f^R(t)$.
After we replace λ_0 in (7) by $\tilde{\lambda}_0(t)$, we can compare the dynamics of the state variables $\mathbf{X}_t$ generated by the constant market price of risk λ_0 (i.e. the case $f = f^R \equiv 0$) and the dynamics for the new state process $\tilde{\mathbf{X}}_t$ generated by the time-varying market price of risk $\tilde{\lambda}_0(t)$:

$$d\mathbf{X}_t^s \;=\; M\left(\mathbb{E}^{\mathbb{Q}}\mathbf{X}_\infty^s - \mathbf{X}_t^s\right)dt \;+\; \Sigma^{r\pi}(\Gamma_0 + (\mathbf{X}_t^s)_1\Gamma)^{\frac{1}{2}}\,dW_t^{\mathbb{Q}}, \tag{27}$$

$$d\tilde{\mathbf{X}}_t^s \;=\; M\left(\mathbb{E}^{\mathbb{Q}}\mathbf{X}_\infty^s - \tilde{\mathbf{X}}_t^s\right)dt \;+\; \Sigma^{r\pi}(\Gamma_0 + (\tilde{\mathbf{X}}_t^s)_1\Gamma)^{\frac{1}{2}}\,dW_t^{\mathbb{Q}} - [0\,1\,0]'f(t)\mathbf{1}_{\{t\geq t_0\}}\,dt, \tag{28}$$

and

$$d\mathbf{X}_t^o \;=\; \left(\mu^o - \left[\begin{smallmatrix}\eta_S\\ \eta_\Pi\end{smallmatrix}\right] + K^o\mathbf{X}_t^s\right)dt \;+\; \Sigma^{S\Pi}(\Gamma_0 + (\mathbf{X}_t^s)_1\Gamma)^{\frac{1}{2}}\,dW_t^{\mathbb{Q}}, \tag{29}$$

$$d\tilde{\mathbf{X}}_t^o \;=\; \left(\mu^o - \left[\begin{smallmatrix}\eta_S\\ \eta_\Pi\end{smallmatrix}\right] + K^o\tilde{\mathbf{X}}_t^s\right)dt \;+\; \Sigma^{S\Pi}(\Gamma_0 + (\tilde{\mathbf{X}}_t^s)_1\Gamma)^{\frac{1}{2}}\,dW_t^{\mathbb{Q}} - [0\,1]'f^R(t)\mathbf{1}_{\{t\geq t_0\}}\,dt. \tag{30}$$

We use the notation $p_{t_0}(\tau)$ and $\tilde{p}_{t_0}(\tau)$ for nominal bond prices generated by the market prices of risk λ_0 and $\tilde{\lambda}_0(t)$ respectively, and $p_{t_0}^R(\tau)$ and $\tilde{p}_{t_0}^R(\tau)$ for the corresponding real bond prices.

We may analyze the effect of the change in the market price of risk using the Riccati equations (78)-(79) in section 10. From those equations and (14)-(15) we conclude that the transformation from λ_0 to $\tilde{\lambda}_0(t)$ leaves Ψ unchanged: $\tilde{\Psi}_{uv}(t,T) = \Psi_{uv}(t,T) = \Psi(T-t)$ and that $\tilde{\zeta}(t) = \zeta(t) - f(t)[0\,1\,0]'$ for $t \geq t_0$, so

$$\ln\frac{\tilde{p}_{t_0}(\tau)}{p_{t_0}(\tau)} \;=\; \tilde{\phi}(t_0,t_0+\tau) - \phi(t_0,t_0+\tau) \;=\; -\int_{t_0}^{t_0+\tau}\Psi_{uv}(s,t_0+\tau)'f(s)[0\,1\,0]'ds$$

$$\partial_\tau\ln\frac{\tilde{p}_{t_0}(\tau)}{p_{t_0}(\tau)} \;=\; -\partial_\tau\int_{t_0}^{t_0+\tau}\Psi(t_0+\tau-s)_2 f(s)ds \;=\; -\int_{t_0}^{t_0+\tau}\dot{\Psi}(t_0+\tau-s)_2 f(s)ds \tag{31}$$

since $\Psi(0)_2 = 0$. Using an approximation which assumes that $\tau \to \ln p_{t_0}(\tau)$ is linear for τ in $[t_0+i\Delta,\, t_0+(i+1)\Delta]$, and since $f(t) = f_i$ on this interval, we find

$$E_i \;:=\; \Delta^{-1}\ln\frac{\tilde{p}_{t_0}((i+1)\Delta)/\tilde{p}_{t_0}(i\Delta)}{p_{t_0}((i+1)\Delta)/p_{t_0}(i\Delta)} \;=\; -\sum_{j=0}^{i}\int_{t_0+j\Delta}^{t_0+(j+1)\Delta}\dot{\Psi}(t_0+(i+1)\Delta-s)_2 f(s)ds \tag{32}$$

$$=\; \sum_{j=0}^{i} f_j\,(\,\Psi((i-j)\Delta) - \Psi((i+1-j)\Delta)\,)_2 \tag{33}$$

which can be used to find the the values of f_i recursively: $f_0 = -E_0/\Psi(\Delta)_2$ and for $i > 0$

$$f_i \;=\; \frac{-E_i}{\Psi(\Delta)_2} \;+\; \sum_{j=0}^{i-1} f_j\frac{\Psi((i-j)\Delta)_2 - \Psi((i+1-j)\Delta)_2}{\Psi(\Delta)_2}. \tag{34}$$

We can then fit the real term structure in a second step using (26). Since $\tilde{\Psi}_{uv}^R(t,T) = \Psi_{uv}^R(t,T) = \Psi^R(T-t)$ we can use, remembering that $\Psi_5^R \equiv 1$,

$$\ln\frac{\tilde{p}_{t_0}^R(\tau)}{p_{t_0}^R(\tau)} \;=\; \tilde{\phi}^R(t_0,t_0+\tau) - \phi^R(t_0,t_0+\tau) \;=\; -\int_{t_0}^{t_0+\tau}\Psi_{uv}^R(s,t_0+\tau)'[\,0\,f(s)\,0\,0\,f^R(s)]'ds$$

$$\partial_\tau\ln\frac{\tilde{p}_{t_0}^R(\tau)}{p_{t_0}^R(\tau)} \;=\; -\int_{t_0}^{t_0+\tau}\dot{\Psi}^R(t_0+\tau-s)_2 f(s)\,ds - f^R(t_0+\tau) \tag{35}$$

so we now find in analogy to the nominal case

$$E_i^R \;:=\; \Delta^{-1}\ln\frac{\tilde{p}_{t_0}^R((i+1)\Delta)/\tilde{p}_{t_0}^R(i\Delta)}{p_{t_0}^R((i+1)\Delta)/p_{t_0}^R(i\Delta)} \tag{36}$$

$$=\; -f_i^R \;+\; \sum_{j=0}^{i} f_j\,(\,\Psi^R((i-j)\Delta) - \Psi^R((i+1-j)\Delta)\,)_2 \tag{37}$$

which can be used to find the values of f_i^R.
When we need to use future (stochastic) discount rates as defined in equations (62)-(63) of subsection 9.3, we can use the following trapezoidal approximations for $i_2 \geq i_1$ which are based on (31) and (35):

$$\begin{aligned}
&\tilde{\phi}(t_0 + i_1\Delta, t_0 + i_2\Delta) - \phi(t_0 + i_1\Delta, t_0 + i_2\Delta) \qquad (38)\\
&= \int_{t_0+i_1\Delta}^{t_0+i_2\Delta} \Psi(t_0 + i_2\Delta - s)'(-f(s)e_2)ds \;=\; \sum_{j=i_1}^{i_2-1} \int_{t_0+j\Delta}^{t_0+(j+1)\Delta} \Psi(t_0 + i_2\Delta - s)'ds\,(-f_j e_2)\\
&\approx \;\Delta \sum_{j=i_1}^{i_2-1} \Big(\frac{\Psi((i_2 - j)\Delta) + \Psi((i_2 - j - 1)\Delta)}{2} \Big)'(-f_j e_2) \qquad (39)
\end{aligned}$$

with $e_2 = [0\,1\,0]'$. Likewise

$$\begin{aligned}
&\tilde{\phi}^R(t_0 + i_1\Delta, t_0 + i_2\Delta) - \phi^R(t_0 + i_1\Delta, t_0 + i_2\Delta) \qquad (40)\\
&\approx \;\Delta \sum_{j=i_1}^{i_2-1} \Big(\frac{\Psi^R((i_2 - j)\Delta) + \Psi^R((i_2 - j - 1)\Delta)}{2} \Big)'(-f_j e_2 - f_j^R e_5),
\end{aligned}$$

with $e_2 = [0\,1\,0\,0\,0]'$ and $e_5 = [0\,0\,0\,0\,1]'$.

7 Constraints

Imposed constraints include the long term average logarithmic annual rate of return on the stock index S and price index Π and the ultimate forward rate[7]:

$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{P}}(\ln S_{t+1} - \ln S_t) \;=\; \mathbb{E}^{\mathbb{P}} r_\infty + \eta_S - \tfrac{1}{2}\sigma_S'(\Gamma_0 + \mathbb{E}^{\mathbb{P}}(\mathbf{X}^s_\infty)_1\Gamma)\sigma_S \;=\; \ln(1 + 0.052) \qquad (41)$$

$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{P}}(\ln \Pi_{t+1} - \ln \Pi_t) \;=\; \mathbb{E}^{\mathbb{P}} \pi_\infty + \eta_\Pi - \tfrac{1}{2}\sigma_\Pi'(\Gamma_0 + \mathbb{E}^{\mathbb{P}}(\mathbf{X}^s_\infty)_1\Gamma)\sigma_\Pi \;=\; \ln(1 + 0.020) \qquad (42)$$

$$\lim_{\tau\to\infty} y_t(\tau) \;=\; \lim_{\tau\to\infty} -\phi(t, t+\tau)/\tau \;=\; \text{UFR} \qquad (43)$$

with the UFR equal to $f_{t_0}(30, 50)$, the nominal forward rate between maturities 30 and 50 year at the time t_0 of calibration, i.e.

$$\text{UFR} = f_{t_0}(30, 50) = \ln\left(\left(\frac{p(t_0,t_0+30)}{p(t_0,t_0+50)}\right)^{1/20}\right) = \frac{50y_{t_0}(50) - 30y_{t_0}(30)}{20} = y_{t_0}(30) + \tfrac{5}{2}(y_{t_0}(50) - y_{t_0}(30)).$$

The limit which determines the nominal UFR equals, for any $t \leq t_0$,

$$\lim_{\tau\to\infty} y_t(\tau) \;=\; -\Psi_\infty{}'\,(\,M\mathbb{E}^{\mathbb{Q}}\mathbf{X}^s_\infty + \tfrac{1}{2}\Sigma^{r\pi}\Gamma_0(\Sigma^{r\pi})'\Psi_\infty\,), \qquad (44)$$

if the vector $\Psi_\infty \in \mathbb{R}^3$ solves the following equation, which follows from equations (15) and (78):

$$-\tfrac{1}{2}\Psi_\infty{}'\Sigma^{r\pi}\Gamma(\Sigma^{r\pi})'\Psi_\infty \mathbf{1}_{i=1} + (M'\Psi_\infty)_i + \mathbf{1}_{i=2} = 0 \qquad (i = 1..3). \qquad (45)$$

We also include a constraint on the expected nominal and real rates 60 years from now for maturity 10 years in equilibrium (i.e. assuming that the state $\mathbf{X}^s$ has converged to its expectation in the long term under $\mathbb{P}$):

$$\tfrac{-1}{10}\left(\tilde{\phi}(60, 70) + \Psi(10)'\mathbb{E}^{\mathbb{P}}[\mathbf{X}^s_\infty]\right) \;=\; \ln(1 + 0.020), \qquad (46)$$

$$\tfrac{-1}{10}\left(\tilde{\phi_R}(60, 70) + \Psi(10)_R'\mathbb{E}^{\mathbb{P}}[\mathbf{X}^s_\infty]\right) \;=\; \ln(1 + 0.000). \qquad (47)$$

[7] Note that the constraint in (43) concerns riskneutral dynamics for asset prices in the past, for which a constant market price of risk λ_0 was assumed.

8 Dynamics of the Dutch Price Index

The consumer price index Π_t that we calibrate in our model concerns the Eurozone Harmonised Index of Consumer Prices (HICP-EU) while Dutch pension funds usually base their decisions on the Dutch Consumer Price Index (CPI-NL) which we indicate by Π_t^{NL}. Statistics Netherlands (CBS) publishes historical observations for this index, $\Pi_{y,m}^{\mathrm{NL,obs}}$, per month m in year y. Average inflation over the yearly period $[y-1, y[$ (i.e. calendar year $y-1$) is approximately[8] the average over the 12 months in that period

$$I_y^{\mathrm{NL}} \approx \sum_{m=1}^{12} \frac{\Pi_{y,m}^{\mathrm{NL,obs}} - \Pi_{y-1,m}^{\mathrm{NL,obs}}}{\Pi_{y-1,m}^{\mathrm{NL,obs}}}. \tag{48}$$

Note this value can only be observed after year y has been completed.
The Netherlands Bureau for Economic Policy Analysis (CPB) publishes forecasts of I_y^{NL} for future calendar years y (or the average over multiple years y in the future). We use this CPI-NL inflation estimate for a future calendar year for all time periods during that year.

For the scenario generator we then assume that inflation in CPI-NL terms over a time interval Δt, $\Pi_{t+\Delta t}^{\mathrm{NL}}/\Pi_t^{\mathrm{NL}}$, equals the inflation in HICP-EU terms, i.e. $\Pi_{t+\Delta t}/\Pi_t$, plus a time-varying spread. The spread is chosen to make the expected values of year-on-year inflation (under $\mathbb{P}$, in logarithmic terms) match the predicted values:

$$\mathbb{E}^{\mathbb{P}}[\ln(\Pi_{t+\Delta t}^{\mathrm{NL}}/\Pi_t^{\mathrm{NL}})] \;=\; \ln(1 + I_{\lfloor t \rfloor + 1}^{\mathrm{NL}})\,\Delta t, \tag{49}$$

with $\lfloor t \rfloor$ the smallest natural number below or equal to t.

At the time of calibration $t_0 = 2022\frac{1}{2}$, the following estimates were available[9]:

I_{2023}^{NL}	I_{2024}^{NL}	I_{2025}^{NL}	$\frac{1}{5}\sum_{t=2026}^{2030} I_t^{\mathrm{NL}}$
0.024	0.024	0.025	0.020

We use the first three CPB estimates for individual years, and for $t = 2026$ up to (and including) $t = 2029$ we use the average value of 0.020 for each one of those years. For later years, we substitute the equilibrium value of 0.020 chosen by the Commissie Parameters. For the current estimates this means that $I_t^{\mathrm{NL}} = 0.020$ for $t \geq 2026$. Since our simulations start in 2022.5, the value of $I_{\lfloor t \rfloor + 1}^{\mathrm{NL}}$ will equal 2.4% for the first 6 months, 2.4% for the 12 months after that, 2.5% for the 12 months after that, and 2.0% from then on.

9 Simulation

After the parameters have been estimated the continuous time dynamics under $\mathbb{P}$ and $\mathbb{Q}$ for $t \geq t_0$:

$$\begin{aligned}
d\begin{bmatrix} \mathbf{X}_t^s \\ \mathbf{X}_t^o \end{bmatrix} &\overset{(2),(4)}{=} \begin{bmatrix} K(\mathbb{E}\mathbf{X}_\infty^s - \mathbf{X}_t^s) \\ \mu^o + K^o \mathbf{X}_t^s \end{bmatrix} dt \;+\; \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} (\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}}\, dW_t^{\mathbb{P}}, \\
&\overset{(7)}{=} \begin{bmatrix} K(\mathbb{E}\mathbf{X}_\infty^s - \mathbf{X}_t^s) \\ \mu^o + K^o \mathbf{X}_t^s \end{bmatrix} dt \;+ \\
&\qquad \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} (\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}} \left(dW_t^{\mathbb{Q}} - ((\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}})^{-1} \left(\tilde{\lambda}_0(t) + \Lambda_1 \mathbf{X}_t^s \right) dt \right), \\
&\overset{(10)-(13)}{\underset{(26)-(30)}{=}} \begin{bmatrix} M(\mathbb{E}^{\mathbb{Q}}\mathbf{X}_\infty^s - \mathbf{X}_t^s) - \begin{bmatrix} 0 \\ f(t) \\ 0 \end{bmatrix} \\ \mu^o + K^o \mathbf{X}_t^s - \begin{bmatrix} \eta_S + 0 \\ \eta_\Pi + f^R(t) \end{bmatrix} \end{bmatrix} dt \;+\; \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} (\Gamma_0 + (\mathbf{X}_t^s)_1 \Gamma)^{\frac{1}{2}} dW_t^{\mathbb{Q}}
\end{aligned} \tag{50}$$

can be used to define discrete simulation schemes under $\mathbb{P}$ and $\mathbb{Q}$.

[8] In fact, the individual months may not be completely uniformly weighted. But that effect turns out to have been negligible in the last few years.
[9] Source: *CPB Raming maart 2022 inclusief Actualisatie Verkenning Middellange Termijn.*

9.1 Simulation under $\mathbb{P}$

For a fixed t_0 we simulate N paths for scenarios

$$\{\,(\mathbf{X}^{s,j}_{t_0+i\Delta t},\,\mathbf{X}^{o,j}_{t_0+i\Delta t},\,R^j_{t_0+i\Delta t})\;=\;(v^j_{t_0+i\Delta t},\,r^j_{t_0+i\Delta t},\,\pi^j_{t_0+i\Delta t},\,\ln S^j_{t_0+i\Delta t},\,\ln\Pi^j_{t_0+i\Delta t},\,R^j_{t_0+i\Delta t})\,\}^{j=1\ldots N}_{i=0\ldots n}$$

each containing n time steps of length $\Delta t = n^{-1}(T_{\max} - t_0)$. Each path starts in known values at time t_0. To simulate a timestep at a time $t = t_0 + i\Delta t$ ($i = 0\ldots n$) we first use Andersen's exact simulation scheme with martingale correction [1] for the Heston model[10] to obtain new values for the v-process, based on iid uniformly distributed samples U^j_t. We then determine the corresponding, approximately Gaussian, increments η^j_t

$$v^j_{t+\Delta t} \;=\; f_{\text{Andersen}}(v^j_t, U^j_t), \qquad \eta^j_t = \omega^{-1}(v^j_t\Delta t)^{-\frac{1}{2}}(v^j_{t+\Delta t} - v^j_t - K_{vv}(\mathbb{E}^{\mathbb{P}}v_\infty - v^j_t)\Delta t). \quad (51)$$

For the remaining state variables we take

$$\begin{bmatrix} r^j_{t+\Delta t} - r^j_t \\ \pi^j_{t+\Delta t} - \pi^j_t \\ \ln(S^j_{t+\Delta t}/S^j_t) \\ \ln(\Pi^j_{t+\Delta t}/\Pi^j_t) \end{bmatrix} \;=\; [0_{4\times1}\;I_4]\left(\begin{bmatrix} K(\mathbb{E}\mathbf{X}^s_\infty - \mathbf{X}^{s,j}_t) \\ \mu^o + K^o\mathbf{X}^{s,j}_t \end{bmatrix}\Delta t \;+\; (\Sigma^{\text{so},j}_t)^{\frac{1}{2}}\begin{bmatrix} \eta^j_t \\ \xi^j_t \end{bmatrix}\right), \quad (52)$$

with

$$\xi^j_t \sim N(0_{4\times1}, I_4)\text{ iid}, \qquad (\Sigma^{\text{so},j}_t)^{\frac{1}{2}} \;=\; \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix}(\Gamma_0 + v^j_t\Gamma)^{\frac{1}{2}}\sqrt{\Delta t}. \quad (53)$$

The integral over the short rate can be updated using $R^j_{t+\Delta t} - R^j_t = r^j_t\Delta t$ and the Dutch consumer price index increment follows from (49):

$$\ln(\Pi^{\text{NL},j}_{t+\Delta t}/\Pi^{\text{NL},j}_t) \;=\; \ln(\Pi^j_{t+\Delta t}/\Pi^j_t) + H^j_t \quad (54)$$

$$H^j_t \;=\; -\frac{1}{N}\sum_{j=1}^{N}\ln(\Pi^j_{t+\Delta t}/\Pi^j_t) \;+\; \ln(1 + I^{\text{NL}}_{\lfloor t\rfloor+1})\,\Delta t. \quad (55)$$

9.2 Simulation under $\mathbb{Q}$

Analogously, we can simulate N paths for scenarios under $\mathbb{Q}$:

$$\{\,(\mathbf{X}^{s,j}_{t_0+i\Delta t},\,\mathbf{X}^{o,j}_{t_0+i\Delta t},\,R^j_{t_0+i\Delta t})\;=\;(v^j_{t_0+i\Delta t},\,r^j_{t_0+i\Delta t},\,\pi^j_{t_0+i\Delta t},\,\ln S^j_{t_0+i\Delta t},\,\ln\Pi^j_{t_0+i\Delta t},\,R^j_{t_0+i\Delta t})\,\}^{j=1\ldots N}_{i=0\ldots n}$$

with the same time steps Δt. Again, we we first use Andersen's exact simulation scheme with martingale correction for the Heston model (but this time under $\mathbb{Q}$) to obtain new values for the v-process, based on iid uniformly distributed samples U^j_t and we determine the corresponding (approximately) Gaussian increments η^j_t

$$v^j_{t+\Delta t} \;=\; f_{\text{Andersen}}(v^j_t, U^j_t), \qquad \eta^j_t = \omega^{-1}(v^j_t\Delta t)^{-\frac{1}{2}}(v^j_{t+\Delta t} - v^j_t - M_{vv}(\mathbb{E}^{\mathbb{Q}}v_\infty - v^j_t)\Delta t). \quad (56)$$

We take $R^j_{t+\Delta t} - R^j_t = r^j_t\Delta t$ and

$$\begin{bmatrix} r^j_{t+\Delta t} - r^j_t \\ \pi^j_{t+\Delta t} - \pi^j_t \\ \ln(S^j_{t+\Delta t}/S^j_t) \\ \ln(\Pi^j_{t+\Delta t}/\Pi^j_t) \end{bmatrix} = [0_{4\times1}\;I_4]\left(\begin{bmatrix} M(\mathbb{E}^{\mathbb{Q}}\mathbf{X}^s_\infty - \mathbf{X}^{s,j}_t) - \left[\begin{smallmatrix} 0 \\ f(t) \\ 0 \end{smallmatrix}\right] \\ \mu^o - \left[\begin{smallmatrix} \eta_S+0 \\ \eta_\Pi+f^R(t) \end{smallmatrix}\right] + K^o\mathbf{X}^{s,j}_t \end{bmatrix}\Delta t + (\Sigma^{\text{so},j}_t)^{\frac{1}{2}}\begin{bmatrix} \eta^j_t \\ \xi^j_t \end{bmatrix}\right) \quad (57)$$

[10] An implementation can be found in the file `MC_QE.m` on the Matlab File Exchange site `nl.mathworks.com/matlabcentral/fileexchange/37618-monte-carlo-simulation-and-derivatives-pricing`.

with

$$\xi_t^j \sim N(0_{4\times 1}, I_4) \text{ iid}, \qquad (\Sigma_t^{\text{so},j})^{\frac{1}{2}} \;=\; \begin{bmatrix} \Sigma^{r\pi} \\ \Sigma^{S\Pi} \end{bmatrix} (\Gamma_0 + v_t^j \Gamma)^{\frac{1}{2}} \sqrt{\Delta t}, \tag{58}$$

and

$$\ln(\Pi_{t+\Delta t}^{\text{NL},\,j}/\Pi_t^{\text{NL},\,j}) \;=\; \ln(\Pi_{t+\Delta t}^j/\Pi_t^j) + H_t^j \tag{59}$$

with H_t^j as defined in (55), i.e. based on the $\mathbb{P}$-scenario's.

9.3 Valuation of derivatives using simulation under $\mathbb{Q}$

We can use the simulated paths to approximate the prices of European call options on the stock index, payer swaptions, zero coupon inflation caps and floors and year-on-year inflation caps and floors that are needed for the calibration. We define nominal and real discount rates at later times $T \geq t_0$

$$D(T, T+\tau) = \mathbb{E}_T^{\mathbb{Q}}[e^{-\int_T^{T+\tau} r_u du}] \;=\; e^{\phi(T,T+\tau)+\Psi(\tau)'\mathbf{X}_T^s}, \tag{60}$$

$$D^R(T, T+\tau) = \mathbb{E}_T^{\mathbb{Q}}[e^{-\int_T^{T+\tau} r_u du} \tfrac{\Pi_{T+\tau}}{\Pi_T}] \;=\; e^{\phi_R(T,T+\tau)+\Psi_R(\tau)'\mathbf{X}_T^s}, \tag{61}$$

and their simulated equivalents

$$\bar{D}(T, T+\tau, \mathbf{X}_T^{s,j}) \;=\; e^{\phi(T,T+\tau)+\Psi(\tau)'\mathbf{X}_T^{s,j}}, \tag{62}$$

$$\bar{D}^R(T, T+\tau, \mathbf{X}_T^{s,j}). \;=\; e^{\phi_R(T,T+\tau)+\Psi_R(\tau)'\mathbf{X}_T^{s,j}}. \tag{63}$$

Derivative prices can then be approximated as follows:

$$\mathbf{C}_{t_0}(T, K) \;=\; \mathbb{E}_{t_0}^{\mathbb{Q}}[e^{-\int_{t_0}^T r_u du}(S_T - K)^+] \tag{64}$$

$$\approx\; \frac{1}{N}\sum_{j=1}^N (e^{-R_T^j}(e^{\ln S_T^j} - K)^+), \tag{65}$$

$$\mathbf{SW}_{t_0}(T_a, T_b, K) \;=\; \mathbb{E}_{t_0}^{\mathbb{Q}}[e^{-\int_{t_0}^{T_a} r_u du}(1 - D(T_a, T_b) - K \sum_{T_k=T_a+1}^{T_b} D(T_a, T_k))^+] \tag{66}$$

$$\approx\; \frac{1}{N}\sum_{j=1}^N e^{-R_{T_a}^j}(\; 1 - \bar{D}(T_a, T_b, \mathbf{X}_{T_a}^{s,j}) - K \sum_{T_k=T_a+1}^{T_b} \bar{D}(T_a, T_k, \mathbf{X}_{T_a}^{s,j})\;)^+ \tag{67}$$

$$\mathbf{YIC}_{t_0}(T, K) \;=\; \sum_{T_k=t_0+1}^{T} \mathbb{E}_{t_0}^{\mathbb{Q}}[e^{-\int_{t_0}^{T_k} r_u du}(\tfrac{\Pi_{T_k} - \Pi_{T_k-1}}{\Pi_{T_k-1}} - K)^+] \tag{68}$$

$$\approx\; \frac{1}{N}\sum_{j=1}^N \sum_{T_k=t_0+1}^{T} e^{-R_{T_k}^j}(e^{\ln \Pi_{T_k}^j - \ln \Pi_{T_k-1}^j} - 1 - K)^+, \tag{69}$$

$$\mathbf{YIF}_{t_0}(T, K) \;=\; \sum_{T_k=t_0+1}^{T} \mathbb{E}_{t_0}^{\mathbb{Q}}[e^{-\int_{t_0}^{T_k} r_u du}(K - \tfrac{\Pi_{T_k} - \Pi_{T_k-1}}{\Pi_{T_k-1}})^+] \tag{70}$$

$$\approx\; \frac{1}{N}\sum_{j=1}^N \sum_{T_k=t_0+1}^{T} e^{-R_{T_k}^j}(K + 1 - e^{\ln \Pi_{T_k}^j - \ln \Pi_{T_k-1}^j})^+, \tag{71}$$

$$\begin{aligned}
\mathbf{IC}_{t_0}(T,K) &= \mathbb{E}^{\mathbb{Q}}_{t_0}[e^{-\int_{t_0}^{T} r_u du}(\tfrac{\Pi_T}{\Pi_{t_0}} - (1+K)^T)^+] &(72)\\
&\approx \frac{1}{N}\sum_{j=1}^{N} e^{-R^j_T}(e^{\ln \Pi^j_T - \ln \Pi_{t_0}} - (1+K)^T)^+, &(73)\\
\mathbf{IF}_{t_0}(T,K) &= \mathbb{E}^{\mathbb{Q}}_{t_0}[e^{-\int_{t_0}^{T} r_u du}((1+K)^T - \tfrac{\Pi_T}{\Pi_{t_0}})^+] &(74)\\
&\approx \frac{1}{N}\sum_{j=1}^{N} e^{-R^j_T}((1+K)^T - e^{\ln \Pi^j_T - \ln \Pi_{t_0}})^+. &(75)
\end{aligned}$$

The implied volatilities and vega values which are needed to specify the goal function in the calibration can be found in the Appendix.

References

[1] L. Andersen. Simple and efficient simulation of the Heston stochastic volatility model. *Journal of Computational Finance*, 11(3):1–42, 2008.

[2] M. J. Brennan and Y. Xia. Dynamic asset allocation under inflation. *The Journal of Finance*, 57(3):1201–1238, 2002.

[3] N. Draper. A financial market model for the Netherlands. Technical report, (CPB Background Document), 2014.

[4] D. Duffie and R. Kan. A yield-factor model of interest rates. *Mathematical Finance*, 6(4):379–406, 1996.

[5] D. Filipovic. *Term-Structure Models: A Graduate Course*. Springer Finance. Springer Berlin Heidelberg, 2009.

[6] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[7] S. Joslin, A. Le, and K. J. Singleton. Why Gaussian macro-finance term structure models are (nearly) unconstrained factor-VARs. *Journal of Financial Economics*, 109(3):604–622, 2013.

[8] R. S. J. Koijen, Th. E. Nijman, and B. J. M. Werker. When can life cycle investors benefit from time-varying bond risk premia? *The Review of Financial Studies*, 23(2):741–780, 2009.

[9] S. Muns. Een financieel marktmodel voor Nederland: een methodologische verfijning. Technical report, (CPB Background Document), 2015.

[10] N. Singor S, L. A. Grzelak, D. D.B. van Bragt, and C. W. Oosterlee. Pricing inflation products with stochastic volatility and stochastic interest rates. *Insurance: Mathematics and Economics*, 52(2):286–299, 2013.

[11] R. Schöbel and J. Zhu. Stochastic volatility with an Ornstein-Uhlenbeck process: an extension. *Review of Finance*, 3(1):23–46, 1999.

[12] A. van Haastrecht and A. Pelsser. Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility. *Quantitative Finance*, 11(5):665–691, 2011.

10 Supplement A
Fourier transform in time-inhomogeneous affine models

If a process X satisfies

$$dX_t = \big(\zeta(t) - LX_t\big)dt + \Sigma\big(G_0 + \sum_i G_i X_t^i\big)^{\frac{1}{2}} dW_t \qquad (76)$$

with W a standard Brownian Motion (under any of the two measure we may wish to consider: $\mathbb{P}$ or $\mathbb{Q}$), then we have for all $t \leq T$ and $t \leq T_1 \leq T_2$

$$\mathbb{E}_t e^{u'X_T + v'\int_t^T X_s ds} = e^{\phi_{uv}(t,T) + \Psi_{uv}(t,T)'X_t}, \qquad (77)$$

if[11]

$$\partial_t \Psi_{uv}(t,T)_i = -\tfrac{1}{2}\Psi_{uv}(t,T)'\Sigma G_i \Sigma' \Psi_{uv}(t,T) + (L'\Psi_{uv}(t,T))_i - v_i, \quad \Psi_{uv}(T,T) = u \qquad (78)$$

$$\phi_{uv}(t,T) = \int_t^T (\, \Psi_{uv}(s,T)'\zeta(s) + \tfrac{1}{2}\Psi_{uv}(s,T)'\Sigma G_0 \Sigma' \Psi_{uv}(s,T)\,)ds. \qquad (79)$$

To check that these Riccati equations are the correct ones, we write

$$Z_t = \mathbb{E}_t e^{u'X_T + v'\int_0^T X_s ds} = e^{\phi_{uv}(t,T) + \Psi_{uv}(t,T)'X_t + v'\int_0^t X_s ds} \qquad (80)$$

and notice that Z is a martingale if a Novikov-style moment condition is satisfied. Applying Itô's lemma to the righthand side gives, using obvious abbreviations

$$d(\ln Z_t) = (\partial_t \phi_{uv} + \partial_t \Psi'_{uv} X_t + v'X_t)dt + \Psi'_{uv} dX_t \qquad (81)$$

$$= (\partial_t \phi_{uv} + \partial_t \Psi'_{uv} X_t + \Psi'_{uv}(\zeta(t) - LX_t) + v'X_t)dt + \Psi'_{uv}\Sigma(G_0 + \sum_i G_i X_t^i)^{\frac{1}{2}} dW_t \qquad (82)$$

which must equal $dZ_t/Z_t - \frac{1}{2}d\langle Z\rangle_t / Z_t^2$ which shows that $dZ_t/Z_t = \Psi'_{uv}\Sigma(G_0 + \sum_i G_i X_t^i)^{\frac{1}{2}} dW_t$ so we must have that

$$-\tfrac{1}{2}\Psi'_{uv}\Sigma(G_0 + \sum_i G_i X_t^i)\Sigma'\Psi_{uv} = \partial_t \phi_{uv} + \Psi'_{uv}\zeta(t) + \sum_i (\partial_t \Psi_{uv} - L'\Psi_{uv} + v)_i X_t^i. \qquad (83)$$

Equating this expression for every component X_t^i, and using the boundary conditions $\Psi_{uv}(T,T) = u$ and $\phi_{uv}(T,T) = 0$, establishes (78) and (79).

11 Supplement B
A parametrization of K and M that ensures positive eigenvalues

The mapping

$$x \mapsto \begin{bmatrix} e^{x_7} & 0 & 0 \\ x_6 & x_1 & x_2 \\ x_5 & x_* & e^{x_3} - x_1 \end{bmatrix}, \qquad x_* = (x_2)^{-1}(x_1(e^{x_3} - x_1) - \tfrac{e^{2x_3}}{1+e^{x_4}})$$

creates matrices K (or riskneutral versions M of K) with positive real eigenvalues for all values $x \in \mathbb{R}^7$ and it is invertible:

$$x_1 = K_{22}, \quad x_2 = K_{23}, \quad x_3 = \ln(T), \quad x_4 = \ln(\tfrac{T^2}{4D} - 1). \quad x_5 = K_{31}, \quad x_6 = K_{21}, \quad x_7 = \ln(K_{11}),$$

with T and D the trace and determinant of the matrix K without its first row and column. The eigenvalues of K are $\lambda_1 = K_{11}$ and $\lambda_{2,3} = \frac{1}{2}T(1 \pm \sqrt{1 - 4D/T^2})$. Our parametrization makes $\lambda_1 = e^{x_7}$ and $\lambda_{2,3} = \frac{1}{2}e^{x_3}(1 \pm 1/\sqrt{1 + e^{-x_4}})$ so positive realness is guaranteed.

[11] Note that the ODEs are formulated in terms of time t; when implemented in terms of time to maturity $\tau = T - t$, a minus sign must be added to the right hand side of the first ODE.

12 Supplement C
Implied volatilities and vega values for derivative instruments

Annualized implied volatilities σ for the different products follow from the equalities

$$\mathbf{C}_{t_0}(T,K) = S_{t_0}\Phi(d_+) - KD(t_0,T)\Phi(d_-), \tag{84}$$

$$d_\pm = \frac{\ln(\frac{S}{KD(t_0,T)}) \pm \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}, \tag{85}$$

$$\mathbf{SW}_{t_0}(T_a,T_b,K) = \left((s_{ab}-K)\,\Phi(\tfrac{s_{ab}-K}{\sigma\sqrt{T_a-t_0}}) + \sigma\sqrt{T_a-t_0}\,\varphi(\tfrac{s_{ab}-K}{\sigma\sqrt{T_a-t_0}})\right)\sum_{T_k=T_a+1}^{T_b} D(t_0,T_k),$$

$$s_{ab} = \frac{D(t_0,T_a)-D(t_0,T_b)}{\sum_{T_k=T_a+1}^{T_b} D(t_0,T_k)}, \tag{86}$$

$$\mathbf{YIC}_{t_0}(T,K) = \sum_{T_k=t_0+1}^{T} D(t_0,T_k)\left(\ \ F_k\Phi(d_{k+})\ \ -(1+K)\Phi(d_{k-})\right), \tag{87}$$

$$\mathbf{YIF}_{t_0}(T,K) = \sum_{T_k=t_0+1}^{T} D(t_0,T_k)\left(-F_k\Phi(-d_{k+}) + (1+K)\Phi(-d_{k-})\right), \tag{88}$$

$$d_{k\pm} = \frac{\ln(\frac{F_k}{1+K}) \pm \frac{1}{2}\sigma^2}{\sigma}, \quad F_k = \frac{D^R(t_0,T_k)/D(t_0,T_k)}{D^R(t_0,T_{k-1})/D(t_0,T_{k-1})}, \tag{89}$$

and

$$\mathbf{IC}_{t_0}(T,K) = D(t_0,T)\left(F\Phi(\tilde{d}_+) - (1+K)^T\Phi(\tilde{d}_-)\right) \tag{90}$$

$$\mathbf{IF}_{t_0}(T,K) = D(t_0,T)\left(-F\Phi(-\tilde{d}_+) + (1+K)^T\Phi(-\tilde{d}_-)\right), \tag{91}$$

$$\tilde{d}_\pm = \frac{\ln(\frac{F}{(1+K)^T}) \pm \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}, \qquad F = \frac{D^R(t_0,T)}{D(t_0,T)}, \tag{92}$$

so the **annualized** vega values $\mho$ equal

$$\frac{\partial \mathbf{C}_{t_0}(T,K)}{\partial\sigma} = S_{t_0}\varphi(d_+)\sqrt{T-t_0}, \tag{93}$$

$$\frac{\partial \mathbf{SW}_{t_0}(T_a,T_b,K)}{\partial\sigma} = \varphi(\tfrac{s_{ab}-K}{\sigma\sqrt{T_a-t_0}})\sqrt{T_a-t_0}\sum_{T_k=T_a+1}^{T_b} D(t_0,T_k), \tag{94}$$

$$\frac{\partial \mathbf{YIC}_{t_0}(T,K)}{\partial\sigma} = \frac{\partial \mathbf{YIF}_{t_0}(T,K)}{\partial\sigma} = \sum_{T_k=t_0+1}^{T} D(t_0,T_k)F_k\varphi(d_{k+}), \tag{95}$$

$$\frac{\partial \mathbf{IC}_{t_0}(T,K)}{\partial\sigma} = \frac{\partial \mathbf{IF}_{t_0}(T,K)}{\partial\sigma} = D^R(t_0,T)\varphi(\tilde{d}_+)\sqrt{T-t_0}. \tag{96}$$

The nominal and real bond prices used to calculate vega values are based on observed yields at the point in time when derivatives prices were quoted, without the UFR. Notice that swaption prices have been expressed in terms of volatilities that correspond to normal, instead of lognormal, distributions.